ON cdh-DESCENT
(*after Haesemeyer et alt.*)

PERE PASCUAL

Abstract. These are the expanded notes of some talks in the working seminar group on Algebraic Geometry at the Universitat Politècnica de Catalunya in 2008. The aim was to understand Haesemeyer's descent arugument, so we present the main results of Haesemeyer and his collaborators on cdh-descent following very closely their papers.

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Date: September 19, 2011.
Partially supported by DGCYT MTM 2009-09557.

1. Introduction

Descent properties of cohomological or homotopical invariants of algebraic varieties over a field of characteristic zero are at the base of many applications of Hironaka's theorem on resolution of singularities. In these notes we study the following situation: let k be a field of characteristic zero and denote by Sch/k the category of k-schemes essentially of finite type over k. Let $F : (Sch/k)^{op} \longrightarrow Sp$ be a contravariant functor with values in the category of spectra.

Question: Under which conditions can we assure that F satisfies descent for (abstract) blow-ups?

More precisely, given a pullback square

$$\begin{array}{ccc} Y' & \xrightarrow{j} & X' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{i} & X \end{array} \tag{Q}$$

with i a closed immersion and p a proper morphism that induces an isomorphism $(X'-Y')_{red} \longrightarrow (X-Y)_{red}$ we look for sufficient conditions on F for the natural morphism

$$F(X) \longrightarrow holim \begin{pmatrix} & & F(X') \\ & & \downarrow{\scriptstyle j^*} \\ F(Y) & \xrightarrow{q^*} & F(Y') \end{pmatrix}$$

to be a weak equivalence of spectra. The conditions we are referring to emerged in the proof of *cdh*-descent of homotopy algebraic K-theory given by Haesemeyer in [H], (see also section 3 of [CHSW]), and are stated as Theorem 3.6.

A closely related question appears when the functor F does not satisfy descent: we can ask if there is a functor F^{abs} which satisfies descent and a morphism $F \longrightarrow F^{abs}$ which is a good enough approximation (in a sense to be defined later). We may cite three approximations to the definition and existence of F^{abs}:

- hyperresolution approach: under certain hypothesis on the action of F on smooth k-schemes, Guillén and Navarro proved in [GN02] that there is a functor $F^{abs} : (Sch/k)^{op} \longrightarrow Ho(Sp)$ which satisfies descent and is unique, in a certain sense, with respect to this property.
- homotopical algebra approach: taking the squares Q as basic coverings of X, we can define a topology on Sch/k. The category of presheaves on this site has a model structure, whose weak equivalences depend on the topology of the site, and Voevodsky proved that the fibrant replacement of a functor F, F^{abs}, satisfies descent, [V10a]. In this case the morphism $F \longrightarrow F^{abs}$ is a local weak equivalence.
- quasi-categorical approach: it is a variation of Voevodsky approach. The descent condition may be thought of as a sheaf condition up to weak equivalence of spectra. The higher topos theory developed by Lurie permits to formalize this notion as sheaves in the $(\infty, 1)$-sites. In this context, F^{abs} is nothing but the sheaf associated to F. This approach has been completed by Roig-Maranges in [RM].

All these approaches are equivalent when they may be compared, (see for example [R09]). One advantage of Guillén-Navarro's approach is that F^{abs} satisfies descent by the very definition and it is given by a finite construction, as $F^{abs}(X)$ is defined as a homotopy limit of a finite diagram of smooth schemes, but at the cost working up to weak equivalence, that is, in the homotopy category $Ho(Sp)$. Moreover, this approach permits to introduce weight filtrations on the homotopy of F^{abs}, (see [GN03], [PR] and also [GS]). In the other hand, Voevodsky's approach has all the homotopical algebra at its disposal producing a functorial fibrant replacement F^{abs} of F, but by a transfinite argument.

We shall present Haesemeyer's argument in [H] that ensures that under certain weak descent hypothesis a functor F satisfies abstract blow-up descent. The proof proceeds by showing that the natural morphism $F(X) \longrightarrow F^{abs}(X)$ is a weak equivalence using an inductive argument and a local to global spectral sequence to reduce the proof to certain local k-schemes X. For the argument to work we shall need that the presheaves of spectra take values in Sp, not only up to weak equivalence, and more importantly that they are defined on schemes *locally* of finite type over k, so we shall follow Haesemeyer paper rather closely and use Voevodsky's approach for the definition of F^{abs}.

We shall sketch some applications of *cdh*-descent to algebraic K-theory and cyclic homology of schemes. These applications are at the base of the solution of Weibel's and Vorst's conjectures, see [CHSW] and [CHW], although we shall be very sketchy at this point, referring to the original papers for the details. We remark that Cortiñas, Haesemeyer, Schlichting, Walker and Weibel have developed further the applications of *cdh*-descent techniques.

Acknowledgements. I want to thank Ll. Rubió and A. Roig-Maranges for his comments and collaboration.

2. Review of Presheaves of Spectra

2.1. **The category of spectra.** We shall work in the category of spectra of simplicial sets, denoted by Sp, taking as main reference the paper by Bousfield-Friedlander [BF]. We consider Sp as (stable) simplicial model category, where a morphism of spectra $X \longrightarrow Y$ is a weak equivalence if and only if the morphism $\pi_*(X) \longrightarrow \pi_*(Y)$ between the stable homotopy groups is an isomorphism.

The fibrant spectra, also called Ω-spectra, are the spectra X_n, $n \geq 0$, with X_n a Kan complex and with structure maps $\Sigma X_n \longrightarrow X_{n+1}$ such that the maps $X_n \longrightarrow \Omega X_{n+1}$, obtained by adjunction, are weak equivalences. As any spectrum has a functorially defined fibrant model, we shall assume implicitly that we work with fibrant spectra.

We shall use the theory of model categories and more specifically the general theory of homotopy limits for simplicial model categories, (see [Hi] or section 5 of Thomason's [T80])). In particular, we shall consider homotopy cartesian diagrams of spectra: that is, commutative diagrams of spectra

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow h \\ A' & \xrightarrow{k} & B' \end{array}$$

which satisfy the equivalent conditions below:

- the natural map from A to the homotopy limit of $A' \xrightarrow{k} B' \xleftarrow{h} B$ is a weak equivalence,
- the map $hofib(A \longrightarrow B) \longrightarrow hofib(A' \longrightarrow B')$ is a weak equivalence,
- the map $hofib(A \longrightarrow A') \longrightarrow hofib(B \longrightarrow B')$ is a weak equivalence,

where $hofib$ denotes the homotopy fiber of a morphism.

We shall also consider bigger homotopy cartesian cubical diagrams of spectra, defined inductively taking homotopy fibers between two faces.

2.2. **Presheaves of spectra on a Grothendieck site.** Let $\mathcal{C}$ be a small category, a presheaf of spectra on $\mathcal{C}$ is a contravariant functor $F : \mathcal{C}^{op} \longrightarrow Sp$. We denote by $Pre(\mathcal{C}, Sp)$ the category of presheaves of spectra on $\mathcal{C}$, with natural transformations as morphisms. As $Pre(\mathcal{C}, Sp)$ is a category of functors, there is a model category structure on it defined levelwise (see [Hi], Theorem 11.7.3). In this structure a morphism of presheaves of spectra $F \longrightarrow G$ is a weak equivalence if for any object $U \in \mathcal{C}$ the induced map of spectra $F(U) \longrightarrow G(U)$ is a weak equivalence. We shall call these morphisms *global weak equivalences.*

If t is a Grothendieck topology on $\mathcal{C}$, there is *local* model structure on $Pre(\mathcal{C}, Sp)$, (in fact, this is one of several model structures with the same weak equivalences). A *local weak equivalence* of presheaves of spectra on $(\mathcal{C}, t)$ is a morphism $f : F \longrightarrow G$ such that the induced morphism $a_t\pi_*(F) \longrightarrow a_t\pi_*(G)$ of sheaves of stable homotopy groups is an isomorphism (where a_tE denotes the sheaf associated to the presheaf E). The following result is due to Jardine, [J87] (as anticipated by Joyal for the sheaf case).

Theorem 2.1. *Let $(\mathcal{C}, t)$ be a Grothendieck site. We consider the following classes of morphisms in the category of presheaves of spectra $Pre(\mathcal{C}, Sp)$:*

- *weak equivalences: the local weak equivalences,*
- *cofibrations: the object wise injections,*
- *fibrations: the morphisms defined by the RLP with respect to trivial cofibrations.*

Then, $Pre(\mathcal{C}, Sp)$ is a proper simplicial model category.

Although we shall not need it, let us remark that fibrant presheaves for this model structure may be characterized after [DHI] by a Cech type condition for hypercoverings: a presheaf F is fibrant if and only if it satisfies the following two conditions:

1. for any $X \in \mathcal{C}$ the spectrum $F(X)$ is fibrant,
2. for any $X \in \mathcal{C}$ and any hypercovering $H \longrightarrow X$ the natural morphism

$$F(X) \longrightarrow hocolim F(H_n)$$

is a weak equivalence of spectra.

We remark the following key property of morphisms between fibrant presheaves, (see [J87], Lemma 2.6, or [CHSW], Section 3).

Proposition 2.2. *A morphism between fibrant presheaves $f : F \longrightarrow G$ is a local weak equivalence if and only if it is a global weak equivalence.*

As a consequence, a fibrant presheaf is globally equivalent to a fibrant sheaf: if F is fibrant presheaf, let a_tF be the associated sheaf, and denote by $(a_tF)^t$ the fibrant replacement of a_tF (in the model category of sheaves), then the morphism $F \longrightarrow (a_tF)^t$ is a local weak equivalence between fibrant presheaves, hence a global weak equivalence.

Any presheaf F has a fibrant model, which may be defined functorially. We denote by F^t the functorial fibrant replacement of F, so that there is a morphism $F \longrightarrow F^t$ which is a local weak equivalence.

Definition 2.3. *We say that a presheaf of spectra F on a site $(\mathcal{C}, t)$* satisfies descent for t *if the local weak equivalence $F \longrightarrow F^t$ is a global weak equivalence.*

That is, F satisfies descent for t if for any $U \in \mathcal{C}$ the morphism $F(U) \longrightarrow F^t(U)$ is a weak equivalence of spectra.

Under finite type conditions on the objects of the site there is a local to global spectral sequence for the fibrant model of a presheaf, (see [MV], Definition 1.31, for the definition of sites of finite type and [J97] for the deduction of the spectral sequence, which follows from the Postnikov tower associated to F^t).

Proposition 2.4. *Let $(\mathcal{C}, t)$ be a Grothendieck site of finite type. For any presheaf of spectra F and any object $X \in \mathcal{C}$ there is a strongly convergent spectral sequence*

$$E_2^{pq} = H_t^p(X, a_t\pi_{-q}F) \Rightarrow \pi_{-p-q}(F^t(X)).$$

In particular, if F satisfies descent for t, then the spectral sequence converges to $\pi_*(F(X))$. We recall that the Zariski, the Nisnevich and the *cdh*-site are finite dimensional.

3. *cd*-structures and the statement of the main result

We recall Voevodsky's notion of *cd*-structure on a category $\mathcal{C}$, [V10a], Definition 2.1.

Definition 3.1. *A cd*-structure *on a category $\mathcal{C}$ is a class $\mathcal{P}$ of commutative squares of $\mathcal{C}$,*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

closed under isomorphism.

If $\mathcal{P}$ and $\mathcal{P}'$ are *cd*-structures, their union $\mathcal{P} \cup \mathcal{P}'$ is also a *cd*-structure, which is called the *combined cd-structure* of $\mathcal{P}$ and $\mathcal{P}'$.

Let k be a field, we write Sch/k for the category of schemes essentially of finite type over k, and Sm/k for the subcategory category of essentially smooth schemes over k. We are interested in some *cd*-structures defined on the category Sch/k, mainly the abstract blow-up structure, the Nisnevich structure and their combination *cdh*, and their restriction to Sm/k. Let's recall their definition, ([V10b], Section 2).

Definition 3.2. *Let*

$$\begin{array}{ccc} Y' & \xrightarrow{j} & X' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{i} & X \end{array} \qquad (Q)$$

be a pull-back square in Sch/k.

(Zar) *Q is an* open (resp. closed) Zariski *square if i, p are open (resp. closed) immersions and $X = i(Y) \cup p(X')$.*

(abs) *Q is an* abstract blow-up square *if i is a closed immersion and p is a proper morphism which induces an isomorphism*

$$(X' - Y')_{red} \longrightarrow (X - Y)_{red}.$$

(Nis) *Q is a* Nisnevich square *if i is an open immersion and p is an étale morphism such that $(X' - Y')_{red} \longrightarrow (X - Y)_{red}$ is an isomorphism.*

(cdh) *Q is a cdh square if it is an abstract or a Nisnevich square.*

Under the name standard *cd*-structure *we shall refer to any of cd-structures on Sch/k defined above.*

Clearly the blow-up of X along Y gives rise to an abstract blow-up square.

If in an abstract blow-up square Q the morphisms p is finite, the square Q will be called a *finite blow-up square.* Observe that a closed covering is a particular case of a finite blow-up square; also an infinitesimal morphism, that is, a morphism $X \longrightarrow Y$ that induces isomorphism between the reduced schemes, $f_{red} : X_{red} \cong Y_{red}$, fits into a finite blow-up square

$$\begin{array}{ccc} \emptyset & \longrightarrow & \emptyset \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

A *cd*-structure $\mathcal{P}$ on a category $\mathcal{C}$ has associated a Grothendieck topology on $\mathcal{C}$, $t_{\mathcal{P}}$: it is the coarsest topology such that for any square of the *cd*-structure the morphisms $\{i : Y \longrightarrow X, p : X' \longrightarrow X\}$ form a covering. If $\mathcal{P}$ is any of the standard *cd*-structures on Sch/k, then it is complete, bounded and regular (see [V10a], Theorem 2.2), so we may characterize the sheaves for $t_{\mathcal{P}}$ as follows, (see [V10a], Corollary 2.17).

Lemma 3.3. *Let $\mathcal{P}$ a standard cd-structure, then a presheaf F is a sheaf for $t_{\mathcal{P}}$ if and only if F takes a distinguished square Q to a pull-back square of spectra $F(Q)$.*

We are interested in a homotopic version of this condition and its relation with the descent property for presheaves.

Definition 3.4. *Let $F : Sch/k \longrightarrow Sp$ a contravariant functor. Given a commutative diagram in Sch/k,*

$$\begin{tikzcd} Y' \arrow[r, "j"] \arrow[d, "q"'] & X' \arrow[d, "p"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$

we say that F satisfies the Mayer-Vietoris property *for this square if the diagram of spectra*

$$\begin{tikzcd} F(X) \arrow[r, "p^*"] \arrow[d, "i^*"'] & F(X') \arrow[d, "j^*"] \\ F(Y) \arrow[r, "q^*"] & F(Y') \end{tikzcd}$$

is homotopy cartesian.

If we equip Sch/k with a cd-structure $\mathcal{P}$, we say that F satisfies MV for $\mathcal{P}$ *if it satisfies the Mayer-Vietoris property for all squares defining $\mathcal{P}$.*

Given a *cd*-structure $\mathcal{P}$ on $\mathcal{C}$ we can consider the local model category of presheaves $Pre(\mathcal{C}, Sp)$ associated to the Grothendieck topology $t_{\mathcal{P}}$.

Theorem 3.5. *Let $\mathcal{P}$ be a standard cd-structure on Sch/k. A presheaf $F \in Pre(Sch/k, Sp)$ satisfies MV property for $\mathcal{P}$ if and only if F satisfies descent for $t_{\mathcal{P}}$.*

Comments on the proof. One implication is easy: if F satisfies descent, it is globally equivalent to a fibrant presheaf and taking the associated sheaf, it is globally equivalent to a fibrant sheaf. If G is a fibrant sheaf and Q a distinguished square then $G(Q)$ is a pullback square by 3.3, but as G is fibrant one can prove that the morphism $G(X) \longrightarrow G(Y)$ is a Kan fibration, so $G(Q)$ being homotopy cartesian is the same as being cartesian, and then the result follows.

The other implication, that a presheaf satisfying MV satisfies descent, is the hard part of Theorem 3.5 and corresponds to a generalization by Voevodsky of Brown-Gesten's main theorem in [BG], see [V10a].

Our objective in this notes is to outline Haesemeyer's proof of the following theorem.

Theorem 3.6. *Let k be a field of characteristic zero and $F : Sch/k \longrightarrow Sp$ a contravariant functor. Suppose that for any smooth k-scheme X the morphism $F(X) \longrightarrow F^t(X)$ is a weak equivalence of spectra and that F is such that:*

(i) F satifies Zariski descent for open and closed coverings,
(ii) F is invariant under infinitesimal extensions,
(iii) F satisfies descent for finite blow-up squares,
(iv) F satisfies descent for regularly embedded blow-ups.

Then, F satisfies descent for abstract blow-ups.

For the combined *cdh*-structure we have:

Corollary 3.7. *Let k be a field of characteristic zero and $F : Sch/k \longrightarrow Sp$ a contravariant functor such that:*

(i) F satisfies excision,
(ii) F is invariant under infinitesimal extensions,
(iii) F satifies Nisnevich descent,
(iv) F satisfies descent for regularly embedded blow-ups.

Then, F satisfies descent for the cdh-structure.

Proof. Let us first observe that for a smooth k-scheme X the morphism $F(X) \longrightarrow F^{cdh}(X)$ is a weak equivalence: the morphism $F \longrightarrow F^{cdh}$ is a local weak equivalence, hence it is also a local weak equivalence when restricted to the *cdh*-site on Sm/k. But on this later site, the restriction of F satisfies descent, by assumptions (iii) and (iv), and the restriction of F^{cdh} satisfies descent, since it satisfies descent for all k-schemes. Hence the restriction to Sm/k of $F \longrightarrow F^{cdh}$ is a local weak equivalence between presheaves that satisfy descent, so it is a global weak equivalence.

In order to apply Theorem 3.6, it only remains to prove that F satisfies descent for finite blow-ups. But this follows from excision, Nisnevich descent and the invariance under infinitesimal extensions. □

The main argument of the proof needs the resolution of singularities, this is the reason of the characteristic zero hypothesis. In fact, we shall need an explicit form of Hironaka's theorem of resolution of singularities, which we recall (see [H64]).

Resolution of singularities 3.8. *Let k be a field of characteristic zero and X an equidimensinoal k-scheme of finite type. Then there is a sequence of morphisms*

$$X_r \longrightarrow \ldots \longrightarrow X_1 \longrightarrow X$$

such that

(1) *the reduced scheme $(X_r)_{red}$ is a smooth k-scheme,*
(2) *each morphism $X_{i+1} \longrightarrow X_i$ in the sequence is the blow-up of X_i along a subscheme Y_i satisfying the following properties:*
   (i) *Y_i is smooth, connected and nowhere dense in X_i,*
   (ii) *X_i is normally flat along Y_i.*

The proof of Theorem 3.6 given by Haesemeyer depends on the results of section 2 and has two main steps:

(1) The passage from a normally flat blow-up to a regularly embedded blow-up.
(2) The proof of the theorem for hypersurfaces on a smooth scheme and, by a Mayer-Vietoris argument, for general k-schemes.

The first point depends on a local algebra result that we shall present in section 4. The final step will be the theme of section 6.

4. An algebraic result

In this section we give an overview of some well known Commutative Algebra results about reduction of ideals used by Haesemeyer in [H]. The objective is to state Proposition 4.14 which, under some hypothesis, gives sufficient conditions for an ideal to be generated by a regular sequence up to an infinitesimal extension. Our main reference for this section is [**?**].

4.1. **Reduction of ideals.** We fix a commutative noetherian ring A and denote by $I, J, \dots$ its ideals.

Definition 4.1. *Let I be an ideal of A. An ideal $J \subset I$ is called a* reduction *of I if there exists a positive integer n_0 such that $JI^n = I^{n+1}$ for any $n \geq n_0$.*

It is easy to prove that if J is a reduction of I then they have the same radical ideal, $\sqrt{J} = \sqrt{I}$, so from the geometrical point of view, a reduction J of an ideal I induces a morphisms of affine schemes $\operatorname{Spec} A/I \longrightarrow \operatorname{Spec} A/J$ which is an infinitesimal extension, that is, such that the morphism of reduced schemes $(\operatorname{Spec} A/I)_{red} \longrightarrow (\operatorname{Spec} A/J)_{red}$ is an isomorphism.

Given an ideal I in A we are interested in finding a reduction J of I which has better embedding properties in A. Here is an illustrative example: let k be a field, $A = k[x, y]$ and consider the ideal $I = (x^2, xy, y^2)$. This ideal defines a non reduced point in the affine plane. Then $J = (x^2, y^2)$ is a reduction of I, since $JI = I^2$, and it is generated by $\{x^2, y^2\}$ which is a regular sequence.

Remark 4.2. The notion of reduction is closely related to that of integral closure of an ideal. Recall that $a \in A$ is *integral over* I if there exist $c_i \in I^i$, $1 \leq i \leq n$, such that

$$a^n + c_1 a^{n-1} + \cdots + c_n = 0.$$

One may prove that $a \in A$ is integral over I if, and only if, I is a reduction of the ideal $I + aA$.

Given an ideal I, we denote by $A[It]$ the graded A-algebra

$$A[It] = A \oplus It \oplus I^2t^2 \oplus \dots$$

If J is a reduction of I then $J^k I^n = I^{k+n}$ for any k and $n \gg 0$, so it is reasonable to expect that this condition imposes some relation between the algebras $A[It]$ and $A[Jt]$, and more concretely between the blow-up of A in I and that of J. The algebraic result is the following.

Proposition 4.3. *$J \subset I$ is a reduction of I if and only if $A[It]$ is a finitely generated $A[Jt]$-module.*

Proof. If J is a reduction of I then

$$A[It]_{k+n} = J^k A[It]_n, \; k \geq 1, \; n \geq n_0,$$

hence the image in $A[It]$ of the finite generators of the A-modules I^i, $0 \leq i \leq n_0$, generate $A[It]$ as an $A[Jt]$-module.

For the converse, let n_0 be the highest degree of a finite set of $A[Jt]$-generators of $A[It]$, then

$$I^{n_0+1}t^{n_0+1} = A[It]_{n_0+1} = \sum (J^i t^i)(I^{n_0+1-i}t^{n_0+1-i}) = JI^{n_0}t^{n_0+1},$$

hence J is a reduction of I. □

From the geometrical point of view we interpret this result as follows.

Corollary 4.4. *Let J be a reduction of I. The the morphism*

$$\mathit{Proj}\, A[It] \longrightarrow \mathit{Proj}\, A[Jt]$$

is a finite morphism.

Proof. For each $x \in J$, it follows from the proposition above that the affine open set $D_+^I(xt) \subset \operatorname{Proj} A[It]$ is finite over the affine open set $D_+^J(xt) \subset \operatorname{Proj} A[Jt]$. Since $(It)^{n+1} \subset (Jt)A[It]$, these open sets form an affine cover of $\operatorname{Proj} A[Jt]$, hence the result. □

Remark 4.5. If A is an integral domain, the converse of Corollary 4.4 is also true, see [W01].

Given an ideal I, we denote by

$$gr(I, A) = \oplus_{n \geq 0} I^n / I^{n+1},$$

the associated graded A-algebra. Reasoning as above, it may be easily proved that if J is a reduction of I, then $gr(I, A)$ is a finitely generated $gr(J, A)$-module.

4.2. **Minimal reductions.** The definition of minimal reduction is clear:

Definition 4.6. *A reduction $J \subset I$ is said to be* minimal *if it does not contain any other proper reduction of I*

In noetherian rings there is no descending chain condition and thus, in general, we can not postulate the existence of minimal reductions of an ideal. However, in local rings minimal reductions do exist.

Proposition 4.7. *Let (A, m) a noetherian local ring with residue field $A/m = k$ and $I \subset A$ an ideal. Let J be a reduction of I, then there exists an ideal $K \subset J$ which is a minimal reduction of I.*

Outline of the proof. Let Σ be the set of all reductions of I contained in J. It is non empty as $J \in \Sigma$.

As A is noetherian, $I \otimes k$ is a finite dimensional k-vector space. We take K in Σ such that $K + mI/mI$ is a subspace of the smallest dimension in $I \otimes k$. It can be proved that K is a reduction of I and by a Nakayama's lemma argument that K is, in fact, a minimal reduction of I. □

There are examples of ideals in a local ring A which do not admit any other reduction than itself, in particular they are their minimal reduction. Let us present an example, referring to [SH] for the details: take $A = \mathbb{Z}/2\mathbb{Z}[[x, y]]/(xy(x + y))$ and $I = m = (x, y)$. It may be proved that if J is a reduction of I, then we can suppose that J is generated by a linear form. But this ring has many few linear forms, concretely x, y and $x + y$. By changing variables one can assume that $J = (x)$, however this ideal is not a reduction of I since $y^{n+1} \notin (x)$.

We observe that if instead of $\mathbb{Z}/2\mathbb{Z}$ we take a larger field k, the ideal I in the example above is not minimal and has as many minimal reductions as units in the field k: let $u \in k$ be a unit, $u \neq 1$, then $J = (x + uy)$ is a reduction of I since

$$\begin{aligned} JI^2 &= (x^3 + ux^2y, x^2y + uxy^2, xy^2 + uy^3) + (xy(x + y)) \\ &= (x^3 + u^2xy^2, xy^2(1 + u), xy^2 + uy^3) + (xy(x + y)) \\ &= (x^3, xy^2, y^3) + (xy(x + y)) = I^3. \end{aligned}$$

This example illustrates that for large fields k there may be many minimal reductions of an ideal, so we expect to have more freedom to chose one with suitable properties. This is the contents of the theorem proved by Northcott-Rees. Before we state it we have to introduce the *spread* of an ideal.

Definition 4.8. *Let (A, m) a noetherian local ring with residue field $A/m = k$ and $I \subset A$ an ideal, we define the* analytic spread *$\ell(I)$ of I as*

$$\ell(I) = \dim gr(I, A) \otimes_A k = \dim A/m \oplus I/mI \oplus I^2/mI^2 \oplus \ldots$$

Remark 4.9. Under certain hypothesis, the analytic spread $\ell(I)$ of an ideal I may be interpreted as the maximum number of elements in I which are analytically independent in I.

Denote by $\mu(I)$ the minimal number of generators of an ideal I. Let J be a reduction of I. As we have remarked, $gr(I, A)$ is a finitely generated $gr(J, A)$-module, and as $gr(J, A)$ is a quotient of $k[x_1, \ldots, x_m]$, with $m = \mu(J)$, we deduce the inequality

$$\mu(J) \leq \ell(I).$$

On a local ring, the equality of this two integers $\mu(J) = \ell(I)$ implies minimality of the reduction J of I and the converse is also true under the hypothesis of having sufficiently many units in the residue field. More concretely, Northcott and Rees proved the following theorem, (see for example [SH]).

Theorem 4.10. *Let (A, m) be a noetherian local ring with infinite residue field $k = A/m$. Then any minimal reduction of an ideal I is generated by exactly $\ell(I)$ elements.* □

Corollary 4.11. *Let (A, m) be a noetherian local ring with infinite residue field $k = A/m$. Then*

$$ht(I) \leq \ell(I) \leq \dim A.$$

Proof. As $gr(I, A) \otimes_A k$ is a quotient of $gr(I, A)$, then $\ell(I) = \dim gr(I, A) \otimes_A k \leq \dim gr(I, A)$ and it is well known that $\dim gr(I, A) \leq \dim A$. If J is a reduction of I, $ht(J) \leq \ell(I)$, and since $ht(J) = ht(I)$, the result follows. □

The hypothesis of having an infinite residue field is not necessary to obtain the inequalities in the corollary above, since they may be proved without appealing to the Northcott-Rees theorem, see [SH].

4.3. **Reductions and regular sequences.** In this paragraph we prove the main result of this section which will enable us to relate, under some hypothesis, descent properties for regular embeddings to descent properties for normally flat subschemes.

Definition 4.12. *The ring A is called* normally flat *along the ideal I if I^n/I^{n+1} is a flat A/I-module, $n \geq 0$.*

That is, A is normally flat along I if the graduated ring defining the normal cone $gr(I, A)$ is a flat A/I-module.

Recall that for a local ring A, I^n/I^{n+1} is A/I-flat if and only if it is a free A/I-module.

Proposition 4.13. *Let (A, m) be a noetherian local ring with infinite residue field $k = A/m$ and suppose it is normally flat along an ideal I. Then I has a minimal reduction J such that*

$$\mu(J) = \ell(I) = ht(I).$$

Proof. Northcott-Rees theorem gives the first equality. As for the second, which is valid even if k is not infinite, observe that as I^n/I^{n+1} is a free A/I-module, for any minimal prime ideal p of I we have

$$\ell_{R_p}(I^nR_p/I^{n+1}R_p) = \ell_R(I^n/mI^n)\ell_{R_p}(R_p/IR_p),$$

hence the degrees of the Hilbert functions $\ell_{R_p}(I^nR_p/I^{n+1}R_p)$ and $\ell_R(I^n/mI^n)$ coincide. But these degrees are $(\ell(I) - 1)$ and $(ht(p) - 1)$, respectively, so $ht(p) = \ell(I)$ for all $p \in Min(I)$, and consequently $ht(I) = \ell(I)$. □

Finally, if we consider Cohen-Macalay rings, the height of an ideal is equal to its depth, so we can relate its analytic spread to its depth.

Proposition 4.14. *Let (A, m) be a noetherian local Cohen-Macaulay ring with infinite residue field $k = A/m$, and let p be a prime ideal such that A is normally flat along p. Then, p has a minimal reduction J generated by a regular sequence of length $\ell(p)$.*

Proof. By Proposition 4.13, p has a reduction J with $\mu(J) = ht(p)$. As p is prime, $\sqrt{J} = p$, hence $ht(p)$ is the codimension of J and therefore J is generated by a regular sequence. □

We can express this result geometrically as follows: the closed immersion $i : \operatorname{Spec} A/I \longrightarrow \operatorname{Spec} A$ factorizes in the form

$$\begin{tikzcd}
\operatorname{Spec} A/I \arrow[r, "i"] \arrow[d, "j"'] & \operatorname{Spec} A \\
\operatorname{Spec} A/J \arrow[ur, "i'"'] &
\end{tikzcd}$$

where j is an infinitesimal extension and i' is a regular embedding.

5. From regular embeddings to normally flat embeddings

In all this section and the following two, we denote by F a presheaf on Sch/k satisfying the hypothesis of Theorem 3.6, that is, it satisfies descent for smooth schemes and properties (i)-(iv).

The theorem of Northcott-Rees, and more specifically Proposition 4.14, enables us to prove that F satisfies descent for formally flat embeddings of integral subschemes of local Cohen-Macaulay schemes.

Proposition 5.1. *Let (A, m) be a noetherian local Cohen-Macaulay ring with infinite residue field $k = A/m$. Let D be an integral subscheme of $S = \operatorname{Spec} A$ along which S is normally flat. Then F satisfies descent for the blow-up of S along D.*

Proof. Consider the blow-up diagram

$$\begin{tikzcd}
D' \arrow[r] \arrow[d] & S_D \arrow[d] \\
D \arrow[r] & S
\end{tikzcd}$$

If p is the prime ideal of D, there is a reduction J of p generated by a regular sequence, by Proposition 4.14. Take $\widetilde{D} = \operatorname{Spec} A/J$ and denote by $S_{\widetilde{D}}$ the blow-up of S along $\widetilde{D}$. By the compatibility of the blow-ups along D and $\widetilde{D}$, there is a commutative diagram

$$\begin{tikzcd}
D' \arrow[rr] \arrow[dr] \arrow[dd] & & S_D \arrow[dr] \arrow[dd, dotted] & \\
& \widetilde{D}' \arrow[rr] \arrow[dd] & & S_{\widetilde{D}} \arrow[dd] \\
D \arrow[rr, dotted] \arrow[dr] & & S \arrow[dr, equal] & \\
& \widetilde{D} \arrow[rr] & & S
\end{tikzcd}$$

whose front and back faces are pullbacks. After applying the functor F, there is a commutative diagram of spectra

$$\begin{tikzcd}
F(D') & & F(S_D) \arrow[ll] & \\
& F(\widetilde{D}') \arrow[ul] \arrow[rr, leftrightarrow] & & F(S_{\widetilde{D}}) \arrow[ul] \\
F(D) \arrow[uu] & & F(S) \arrow[ll, dotted] \arrow[uu, dotted] \arrow[dr, equal] & \\
& F(\widetilde{D}) \arrow[ul] \arrow[uu] & & F(S) \arrow[ll] \arrow[uu]
\end{tikzcd}$$

We want to prove that the back face is homotopy cartesian. Since F satisfies descent for regularly embedded blow-ups, by (iv), the front face is homotopy cartesian, so the result will follow if we can prove that the entire cube is homotopy cartesian. For that, we shall see that the top and bottom faces are homotopy cartesian. The bottom face is homotopy cartesian because F is invariant under infinitesimal extensions, (ii).

As for the top face, let $\widetilde{D}'' = \widetilde{D}' \times_{S_{\widetilde{D}}} S_D$. The square

$$\begin{tikzcd} \widetilde{D}'' \arrow[r] \arrow[d] & S_D \arrow[d] \\ \widetilde{D}' \arrow[r] & S_{\widetilde{D}} \end{tikzcd}$$

is a finite blow-up square, so by (iii) it induces an homotopy cartesian square after applying F. Hence, to conclude the proof it only remains to observe that the morphism $D' \longrightarrow \widetilde{D}''$ is an infinitesimal extension, and to apply once more time that F is invariant under infinitesimal extensions, (ii), so that $F(\widetilde{D}'') \longrightarrow F(D')$ is a weak equivalence. $\square$

By sheafifying F and using the local to global spectral sequence for the Zariski topology, we can extend the local result just proved as follows, (recall that k has characteristic zero, so it is infinite).

Theorem 5.2. *Let X be a Cohen-Macaulay k-scheme and $Y \subset X$ an integral subscheme along which X is normally flat. Then, under the hypothesis of 3.6, F satisfies descent with respect to the blow-up of X along Y.*

Proof. Let X' be the blow-up of X along Y, and Y' the exceptional divisor. Consider the presheaves for the open sets of X defined by

$$\begin{aligned} F_1(U) &= hofib(F(U) \longrightarrow F(X' \times XU), \\ F_2(U) &= hofib(F(Y \times_X U) \longrightarrow F(Y' \times XU). \end{aligned}$$

Observe that these presheaves satisfy Zariski descent by construction.

By Proposition 5.1, the morphism $F_1 \longrightarrow F_2$ is a local weak equivalence, hence from the descent spectral sequence and Zariski descent it follows that $F_1(X) \longrightarrow F_2(X)$ is a weak equivalence. $\square$

6. Continuation of the proof of the main result

Recall that F denotes a presheaf satisfying the hypothesis of Theorem 3.6.

6.1. **Hypersurfaces of smooth schemes.** We want to prove that for any k-scheme X the natural morphism $F(X) \longrightarrow F^{abs}(X)$ is a weak equivalence. We begin by proving this result for hypersurfaces in smooth schemes.

Proposition 6.1. *Let X by an hypersurface of a smooth k-scheme U. Then, $F(X) \longrightarrow F^{abs}(X)$ is a weak equivalence.*

Proof. We use induction on the dimension of X. For $\dim X = 0$ there is nothing to prove, so we may assume that the result holds for hypersurfaces of dimension $n-1$ of smooth schemes and consider the case where $\dim X = n$.

By Hironaka's theorem on resolution of singularities, there is a sequence of admissible blow-ups

$$X_r \longrightarrow \cdots \longrightarrow X_0 = X,$$

such that X_r is smooth. We use induction on the minimal length $s(X)$ of such a sequence.

If $s(X) = 0$, then X is already smooth, so $F(X) \longrightarrow F^{abs}(X)$ is a weak equivalence by hypothesis. Assume now that $s = s(X) > 0$ and take a minimal sequence of length s,

$$X_s \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = X.$$

Let Y be the smooth center of the blow-up $X_1 \longrightarrow X$ and Y' the exceptional divisor. By application of F and F^{abs} to the blow-up diagram we obtain a commutative diagram of spectra

$$\begin{tikzcd}
F(X) \arrow[rr] \arrow[dr] \arrow[dd] & & F^{abs}(X) \arrow[dr] \arrow[dd, dotted] & \\
& F(X_1) \arrow[rr] \arrow[dd] & & F^{abs}(X_1) \arrow[dd] \\
F(Y) \arrow[rr, dotted] \arrow[dr] & & F^{abs}(Y) \arrow[dr] & \\
& F(Y') \arrow[rr] & & F^{abs}(Y')
\end{tikzcd}$$

By Theorem 5.2, the left face is homotopy cartesian, while the right face is homotopy cartesian because F^{abs} satisfies abstract blow-up descent. Hence the hole diagram is homotopy cartesian.

Now, let us analyze the horizontal arrows, if we prove that the horizontal arrows corresponding to Y, Y' and X_1 are weak equivalences, then the result will follow because the diagram is homotopy cartesian.

- (Y) the morphism $F(Y) \longrightarrow F^{abs}(Y)$ is a weak equivalence since Y is smooth.
- (X_1) let us first observe that X_1 is an hypersurface of the blow-up of U along Y, $Bl_Y(U)$, and that this blow-up is smooth. Now, $F(X_1) \longrightarrow F^{abs}(X_1)$ is a weak equivalence by induction hypothesis, since $s(X_1) < s$.
- (Y') let Y'' be the exceptional divisor of the blow-up $Bl_Y(U)$, which is smooth. Then Y' is a Cartier divisor on Y'', hence it is an hypersurface in a smooth scheme, and its dimension is $n-1$, so the morphism $F(Y') \longrightarrow F^{abs}(Y')$ is a weak equivalence by induction on dimension.

□

Now we can generalize this result to complete intersections on a smooth scheme.

Corollary 6.2. *Let X be a local complete intersection on a smooth scheme U. Then the morphism $F(X) \longrightarrow F^{abs}(X)$ is a weak equivalence.*

Proof. By Zariski descent, (i), the question is local, so we can assume that X is a complete intersection on U, defined by a regular sequence $(f_1, \ldots, f_c)$. We proceed by induction on the codimension c.

For $c = 1$, the result is true by Proposition 6.1. For $c > 1$, consider X as the intersection of two complete intersections of codimension $c - 1$,

$$X = V(f_1, f_3, \ldots, f_c) \cap V(f_2, f_3, \ldots, f_c),$$

and observe that their union, that is the subscheme defined by $(f_1 f_2, f_3, \ldots, f_c)$, is also a complete intersection in U of codimension $c - 1$. Finally apply the induction hypothesis and descent for closed coverings, (i). $\square$

6.2. **The final step.** Now we can prove that $F(X) \longrightarrow F^{abs}(X)$ is a weak equivalence for any k-scheme X, where k is a field of characteristic zero, using a Mayer-Vietoris argument for closed coverings. We always assume that F is a presheaf satisfying the hypothesis of Theorem 3.6.

Theorem 6.3. *Let X be a k-scheme, then $F(X) \longrightarrow F^{abs}(X)$ is a weak equivalence. In particular, F satisfies abstract blow-up descent.*

Proof. Once more we prove the result by induction on the dimension of X.

We first observe that we can assume that X is an integral local k-scheme: the integrality assumption follows from descent for closed coverings and invariance for infinitesimal extensions, (i), (ii), while we can suppose that X is local by descent for Zariski open coverings, (i).

Remark that X being local can be embedded into a smooth connected local scheme U in such a way that there is a complete intersection $W \subset U$ such that X is one of its components: in fact, let A be the regular local ring of U and p the prime ideal defining X, then there is a regular sequence $\{f_1, \ldots, f_c\} \subset p$, where $c = ht(p)$, over which p is minimal, so we can take $W = V(f_1, \ldots, f_c)$.

Write X' for the (reduced) union of the other components of the complete intersection W, that is, X' is the closure in W of $W - X$, and take $Y = X \cap X''$. We get a closed covering square

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X' \arrow[d] \\ X \arrow[r] & W^{red} \end{tikzcd} \qquad (P)$$

which after applying F and F^{abs} gives homotopy cartesian squares by descent for closed coverings, (ii). As F, F^{abs} are invariant under infinitesimal extensions, (ii), they induce homotopy cartesian squares associated to (P) and, consequently, distinguished triangles in $Ho(Sp)$, so the morphism $F \longrightarrow F^t$ induces a morphism of triangles

$$\begin{tikzcd} F(W) \arrow[r] \arrow[d] & F(X) \oplus F(X') \arrow[r] \arrow[d] & F(Y) \arrow[d] \\ F^{abs}(W) \arrow[r] & F^{abs}(X) \oplus F^{abs}(X') \arrow[r] & F^{abs}(Y) \end{tikzcd}$$

Observe that the dimension of Y is less than the dimension of X, so we can use the induction hypothesis to ensure that the right morphism is a weak equivalence. By Corollary 6.2, the result is also true for the complete intersection W, hence the result follows from the five lemma and additivity. □

7. Examples and applications

In this section we apply the main theorem and its corollary to the algebraic K-theory and the cyclic homology of k-scheme X. The application to algebraic K-theory in [H] was inspired by the calculus of the algebraic K-theory of a regularlly embedded subscheme by Thomason, **??**. We shall follow [CHSW] in order to deduce both applications from the localization property of algebraic K-theory and cyclic homology.

Let X be a (quasi-compact and quasi-separated) scheme. Denote by $D_{per}(X)$ the derived category of perfect complexes of X, [TT]. Let $i : Y \longrightarrow X$ be a regular embedding of pure codimension d and

$$\begin{tikzcd} Y' \arrow[r, "j"] \arrow[d, "q"'] & X' \arrow[d, "p"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$

the associated blow-up square. For $\ell = 0, \ldots, d-1$, consider the following triangulated categories:

$D^{\ell}_{per}(X')$: the full triangulated subcategory of $D_{per}(X')$ generated by Lp^*F and $Rj_*Lq^*G \otimes \mathcal{O}(-k)$ for $F \in D_{per}(X)$, $G \in D_{per}(Y)$ and $k = 1, \ldots, \ell$.

$D^{\ell}_{per}(Y')$: the full triangulated subcategory of $D_{per}(Y')$ generated by $Lq^*G \otimes \mathcal{O}(-k)$ for $G \in D_{per}(Y)$ and $k = 0, \ldots, \ell$.

The following result may be deduced by a clever interpretation of the classical results on perfect complexes of a blow-up in [SGA6], see [CHSW] and [T93].

Proposition 7.1. *The functor Lj^* is compatible with the filtrations on $D_{per}(X')$ and $D_{per}(Y')$:*

$$\begin{tikzcd} D_{per}(X) \arrow[r, "Lp^*", "\sim"'] \arrow[d, "Li^*"] & D^0_{per}(X') \arrow[d, "Lj^*"] & \subseteq & D^1_{per}(X') \arrow[d, "Lj^*"] & \subseteq & \cdots & \subseteq & D^{d-1}_{per}(X') = D_{per}(X') \arrow[d, "Lj^*"] \\ D_{per}(Y) \arrow[r, "Lp^*", "\sim"'] & D^0_{per}(Y') & \subseteq & D^1_{per}(Y') & \subseteq & \cdots & \subseteq & D^{d-1}_{per}(Y') = D_{per}(Y'). \end{tikzcd}$$

For $\ell = 0, \ldots, d-2$, Lj^* induces equivalences in the triangulated quotient categories

$$Lj^* : D^{\ell+1}_{per}(X')/D^{\ell}_{per}(X') \xrightarrow{\sim} D^{\ell+1}_{per}(Y')D^{\ell}_{per}(Y').$$

7.1. **Homotopy algebraic K-theory.** We denote by K the presheaf on the category of schemes of nonconnective K-theory spectra as defined by Thomason in [TT], Definition 6.4, and by $KH(X)$ the homotopy invariant K-theory introduced by Weibel in [W89a], as reformulated by Thomason, [TT], Exercise 9.11. There is a natural transformation $K \longrightarrow KH$.

Theorem 7.2. *The homotopy algebraic K-theory KH of schemes essentially of finite type over a field of characteristic zero satisfies cdh-descent.*

Proof. Let us remark that KH satisfies hypothesis (i)-(iv) of Corollary 3.7, so the theorem will follow:

(i) KH satisfies excision by [W89a], Theorem 2.1, (see also [TT], Exercise 9.11).

(ii) KH satisfies invariance under infinitesimal extensions by [W89a], Theorem 2.3 (see also [TT], Exercise 9.11 (e)).

(iii) Descent for finite blow-ups for KH is proved in [W89a], Proposition 4.9.

(iv) Regularly embedded blow-ups: Thomason has calculeted in [T93] Th'eorème 2.1, the algebraic K-theory of a blow-up along a regularly embedded subscheme. From this explicit calculation it easily follows that K-theory satisfies MV for a cartesian square coming from regularly embedded subscheme $Y \subset X$. This result follows also directly from Proposition 7.1 by iterated application of Thomason's localization theorem ([TT]) for each of the cartesian squares there.

Now, for any n, the embedding $Y \times \Delta^n \longrightarrow X \times \Delta^n$ is also regular, so by Thomason's result we obtain a homotopy cartesian square

$$\begin{array}{ccc} K(X \times \Delta^n) & \longrightarrow & K(Y \times \Delta^n) \\ \downarrow & & \downarrow \\ K(X' \times \Delta^n) & \longrightarrow & K(Y' \times \Delta^n) \end{array}$$

that is also homotopy cocartesian, since we are working in the stable category of spectra. But then, taking homotopy colimits we obtain a homotopy cocartesian square

$$\begin{array}{ccc} KH(X) & \longrightarrow & KH(Y) \\ \downarrow & & \downarrow \\ KH(X') & \longrightarrow & KH(Y') \end{array}$$

which consequently is a homotopy cartesian square. $\square$

Corollary 7.3. *The natural morphism $K \longrightarrow KH$ induces an equivalence $K^{cdh} \xrightarrow{\sim} KH$.*

Remark 7.4. Cisinski has proved in [Ci] that KH satisfies *cdh*-descent by proving a result enounced by Voevodsky, which states the representability of KH by the $S^1 \wedge \mathbb{G}_m$-spectrum KGL on the stable homotopy category of schemes defined by Morel and Voevodsky, and using smooth base change theorems in the motivic context. The advantage of Cisinski's proof is that it does not use resolution of singularities, hence the result follows also for schemes over a field of finite characteristic.

7.2. **Cyclic homology.** In this paragraph we shall work with complexes of abelian groups rather than with spectra. Using the Dold-Kan correspondence it is easy to argue that Theorem 3.5 remains valid in this context.

Recall that a mixed complex (C, b, B) is a cochain complex of abelian groups (C, b) together with a chain map $B : C \longrightarrow C[-1]$ satisfying $B^2 = 0$ and $Bb + bB = 0$. There is an evident notion of morphism of mixed complexes. We denote by $\mathcal{M}ix$ the category of mixed complexes and by $D\mathcal{M}ix$ the category obtained from $\mathcal{M}ix$ by inverting the quasi-isomorphisms.

Given a mixed complex (C, b, B), we can use the B operator to define some naturally associated double complexes and their corresponding (product-) total complexes, giving rise to $HC(C), HP(C), HN(C)$, that is the cyclic homology, the periodic cyclic homology and the negative cyclic homology complexes associated to C,

$$\begin{aligned} HC(C) &= \mathrm{Tot}(\cdots \xrightarrow{B} C[-1] \xrightarrow{B} C \longrightarrow 0 \longrightarrow \dots), \\ HP(C) &= \mathrm{Tot}(\cdots \xrightarrow{B} C[-1] \xrightarrow{B} C \xrightarrow{B} C[1] \xrightarrow{B} \dots), \\ HN(C) &= \mathrm{Tot}(\dots \longrightarrow 0 \xrightarrow{B} C \xrightarrow{B} C[1] \xrightarrow{B} \dots). \end{aligned}$$

For any mixed complex C, theses complexes are related by a natural exact sequence

$$0 \longrightarrow HN(C) \longrightarrow HP(C) \longrightarrow HC(C)[2] \longrightarrow 0,$$

which defines a triangle in the derived category of complexes of abelian groups.

Keller associates a mixed complex $C(\mathcal{A})$ to any dg-category $\mathcal{A}$ (see [K06] and the cites therein for a review of dg-categories and the definition of $C(\mathcal{A})$). This applies to schemes as follows: let X be a k-scheme, we denote by $Per_{dg}(X)$ the dg-category of perfect complexes on X and define the mixed complex $C(X) = C(Per_{dg}(X))$. We write $HC(X), HP(X), HN(X)$ for the cyclic, periodic cyclic and negative cyclic homology of the mixed complex $C(X)$.

Theorem 7.5. *The periodic cyclic homology HP of schemes essentially of finite type over a field of characteristic zero satisfies cdh-descent.*

Proof. We analize assumptions (i)-(iv) of Corollary 3.7:

(i) The excision property for periodic cyclic homology is a result proved by Cuntz and Quillen in [CQ], Theorem 5.3.

(ii) Goodwillie proved in [G85], Theorem II.5.1, that HP is invariant under infinitesimal extensions.

(iii) The Nisnevich descent for HP follows from the étale descent proved in [GRW], (see also [CHSW], Theorem 2.9).

(iv) Regularly embedded descent: Keller has proved a localization theorem for the mixed complex associated to localization pairs of dg-categories which is the analogue of Thomason's localization theorem for algebraic K-theory, (see [K99], Theorem 2.4). From this we can deduce the MV property for regularly embedded blow-up squares after Proposition 7.1, as we did for K-theory: each square induces homotopy cartesian squares of mixed complexes, hence the result. $\square$

Remark 7.6. The same proof as the one given for periodic cyclic homology gives that the cyclic and negative cyclic homology functors, HC, HN, defined on the category of schemes essentially of finite type over a field of characteristic zero satisfy the Mayer-Vietoris property for regular blow-up squares. The same references as in the proof above give that they also satisfy descent for Nisnevich covers.

7.3. **The fiber of $F \longrightarrow F^{cdh}$ for some functors.** Let $F : Sch/k \longrightarrow Sp$ be a presheaf of spectra and F^{cdh} its fibrant model for the *cdh*-topology. We denote by $\mathcal{F}_F$ the homotopy fiber of the natural morphism $F \longrightarrow F^{cdh}$, so we have a homotopy fiber sequence

$$\mathcal{F}_F \longrightarrow F \longrightarrow F^{cdh}.$$

The following remark will be useful in the sequel.

Lemma 7.7. *Let $E \longrightarrow F \longrightarrow G$ be a homotopy fibration sequence of presheaves of spectra. Then there is a natural induced homotopy fibration sequence*

$$\mathcal{F}_E \longrightarrow \mathcal{F}_F \longrightarrow \mathcal{F}_G.$$

We observe also that if F satisfies one of the hypothesis in Corollary 3.7, then $\mathcal{F}_F$ satisfies this property too. For example, if F in invariant under infinitesimal extensions, as F^{cdh} is also invariant, it follows immediatly that $\mathcal{F}_F$ is invariant under infinitesimal extensions. In particular, it follows from the results above that $\mathcal{F}_K$ and $\mathcal{F}_{HN}$ satisfy descent for Nisnevich and regularly embedded blow-ups squares.

Examples 7.8. 1. For any scheme X there is a Chern character morphism $K(X) \longrightarrow HN(X)$ defining a natural transformation of functors $K \longrightarrow HN$, (see [CHSW], Section 4). The homotopy fiber of this morphism is the presheaf known as infinitesimal K-theory, K^{inf}, so there is a homotopy fibration sequence $K^{inf} \longrightarrow K \longrightarrow HN$. From the lemma we obtain a homotopy fibration sequence

$$\mathcal{F}_{K^{inf}} \longrightarrow \mathcal{F}_K \longrightarrow \mathcal{F}_{HN}.$$

2. The homotopy fibration sequence $HN \longrightarrow HP \longrightarrow HC[2]$ induces a homotopy fibration sequence

$$\mathcal{F}_{HN} \longrightarrow \mathcal{F}_{HP} \longrightarrow \Omega^{-2}\mathcal{F}_{HC}$$

but HP satisfies *cdh*-descent, $\mathcal{F}_{HP} \sim *$ and it follows that there is a natural equivalence

$$\mathcal{F}_{HN} \cong \Omega^{-1}\mathcal{F}_{HC}.$$

The infinitesimal K-theory K^{inf} has been studied by Cortiñas in [C].

Theorem 7.9. *The infinitesimal K-theory, K^{inf}, satisfies cdh-descent.*

Proof. As for KH and HP we give the references which assure that K^{inf} satisfies the assumptions of Corollary 3.7: The excision follows from the properties proved by Cortiñas in [C]; the infinitesimal invariance follows from Goodwillie's Theorem II.3.4 in [G86]. Finally, the Nisnevich and regularly embedded blow-up descent follow from the observation before the examples above. □

Corollary 7.10. *For any k-scheme X, the Chern character $K \longrightarrow HN$ induces natural weak equivalences*

$$\mathcal{F}_K(X) \xrightarrow{\sim} \mathcal{F}_{HN}(X) \xleftarrow{\sim} \Omega^{-1}\mathcal{F}_{HC}(X).$$

7.4. **Final rermarks.** Cortiñas, Haesemeyer, Schlichting and Weibel have used the abstract results above to prove Weibel's conjecture and Vorst's conjecture for schemes over a field of characteristic zero, [CHSW], [CHW]. Let us give the results, referring to the original papers for the proofs.

Recall that a scheme X is called K_n-regular if the morphisms $K_n(X) \longrightarrow K_n(X \times \mathbb{A}^r)$ are isomorphisms for any r and that if X is K_n-regular, then it is also K_{n-1}-regular.

Theorem 7.11 (Weibel's conjecture, [CHSW], Theorem 6.2)**.** *Let k be a field of characteristic zero and X a k-scheme of dimension d. Then*

$$K_m(X) = 0, \qquad \textit{for} \quad m < -d,$$

and X is K_{-d}-regular.

Theorem 7.12 (Vorst's conjecture, [CHW], Theorem 0.1 (c))**.** *Let A be a commutative ring essentially of finite type over a field k of characteristic zero and $d = \dim A$. If A is K_{d+1}-regular, then A is regular.*

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(P. Pascual) Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona (Spain).

E-mail address: pere.pascual@upc.es